- For a sequence x with Fourier transform X, the following assertions hold:
  1. x is even $\Leftrightarrow X$ is even; and
  2. x is odd $\Leftrightarrow X$ is odd.
- In other words, the forward and inverse Fourier transforms preserve even/odd symmetry.

- A sequence x is *real* if and only if its Fourier transform X satisfies

$$X(\Omega) = X^*(-\Omega) \quad \text{for all } \Omega$$

  (i.e., X has *conjugate symmetry*.)
- Thus, for a real-valued sequence, the portion of the graph of a Fourier transform for negative values of frequency Ω is *redundant*, as it is completely determined by symmetry.
- From properties of complex numbers, one can show that $X(\Omega) = X^*(-\Omega)$ is equivalent to

$$|X(\Omega)| = |X(-\Omega)| \quad \text{and} \quad \arg X(\Omega) = -\arg X(-\Omega)$$

  (i.e., $|X(\Omega)|$ is *even* and $\arg X(\Omega)$ is *odd*.)
- Note that x being real does *not* necessarily imply that X is real.

- The DTFT analysis and synthesis equations are, respectively, given by

$$X(\Omega) = \sum_{k=-\infty}^{\infty} x(k) e^{-jk\Omega} \quad \text{and} \quad x(n) = \frac{1}{2\pi} \int_{2\pi} X(\Omega) e^{jn\Omega} d\Omega.$$

- The CTFS synthesis and analysis equations are, respectively, given by

$$x_c(t) = \sum_{k=-\infty}^{\infty} a(k) e^{jk(2\pi/T)t} \quad \text{and} \quad a(n) = \frac{1}{T} \int_{T} x_c(t) e^{-jn(2\pi/T)t} dt,$$

  which can be rewritten, respectively, as

$$x_c(t) = \sum_{k=-\infty}^{\infty} a(-k) e^{-jk(2\pi/T)t} \quad \text{and} \quad a(-n) = \frac{1}{T} \int_{T} x_c(t) e^{jn(2\pi/T)t} dt.$$

- The CTFS synthesis equation with $T = 2\pi$ corresponds to the DTFT analysis equation with $X = x_c$, $\Omega = t$, and $x(n) = a(-n)$.
- The CTFS analysis equation with $T = 2\pi$ corresponds to the DTFT synthesis equation with $X = x_c$ and $x(n) = a(-n)$.
- Consequently, the DTFT X of the sequence x can be viewed as a CTFS representation of the 2π-periodic spectrum X.

- The Fourier transform can be generalized to also handle periodic signals.
- Consider an N-periodic sequence x.
- Define the sequence x_N as

$$x_N(n) = \begin{cases} x(n) & \text{for } 0 \le n < N \\ 0 & \text{otherwise.} \end{cases}$$

(i.e., $x_N(n)$ is equal to $x(n)$ over a single period and zero elsewhere).

- Let a denote the Fourier series coefficient sequence of x.
- Let X and X_N denote the Fourier transforms of x and x_N, respectively.
- The following relationships can be shown to hold:

$$X(\Omega) = \frac{2\pi}{N} \sum_{k=-\infty}^{\infty} X_N\left(\frac{2\pi k}{N}\right) \delta\left(\Omega - \frac{2\pi k}{N}\right),$$

$$a_k = \frac{1}{N} X_N\left(\frac{2\pi k}{N}\right), \quad \text{and} \quad X(\Omega) = 2\pi \sum_{k=-\infty}^{\infty} a_k \delta\left(\Omega - \frac{2\pi k}{N}\right).$$

- The Fourier series coefficient sequence a is produced by sampling X_N at integer multiples of the fundamental frequency $\frac{2\pi}{N}$ and scaling the resulting sequence by $\frac{1}{N}$
- The Fourier transform of a periodic sequence can only be nonzero at integer multiples of the fundamental frequency.

Section 10.4

Fourier Transform and Frequency Spectra of Signals

- Like Fourier series, the Fourier transform also provides us with a frequency-domain perspective on sequences.
- That is, instead of viewing a sequence as having information distributed with respect to *time* (i.e., a function whose domain is time), we view a sequence as having information distributed with respect to *frequency* (i.e., a function whose domain is frequency).
- The Fourier transform X of a sequence x provides a means to *quantify* how much information x has at different frequencies.
- The distribution of information in a sequence over different frequencies is referred to as the *frequency spectrum* of the sequence.

- To gain further insight into the role played by the Fourier transform X in the context of the frequency spectrum of x, it is helpful to write the Fourier transform representation of x with $X(\Omega)$ expressed in *polar form* as follows:

$$x(n) = \frac{1}{2\pi}\int_{2\pi} X(\Omega)\,e^{j\Omega n}d\Omega = \frac{1}{2\pi}\int_{2\pi} |X(\Omega)|\,e^{j[\Omega n+\arg X(\Omega)]}d\Omega.$$

- In effect, the quantity $|X(\Omega)|$ is a *weight* that determines how much the complex sinusoid at frequency Ω contributes to the integration result $x(n)$.
- Perhaps, this can be more easily seen if we express the above integral as the *limit of a sum*, derived from an approximation of the integral using the area of rectangles, as shown on the next slide. [Recall that $\int_a^b f(x)\,dx = \lim_{n\to\infty}\sum_{k=1}^{n} f(x_k)\Delta x$ where $\Delta x = \frac{b-a}{n}$ and $x_k = a + k\Delta x$.]

- Expressing the integral (from the previous slide) as the *limit of a sum*, we obtain

$$x(n) = \lim_{\ell \to \infty} \frac{1}{2\pi} \sum_{k=1}^{\ell} \Delta\Omega \left|X(\Omega')\right| e^{j[\Omega' n + \arg X(\Omega')]},$$

  where $\Delta\Omega = \frac{2\pi}{\ell}$ and $\Omega' = k\Delta\Omega$.
- In the above equation, the kth term in the summation corresponds to a complex sinusoid with fundamental frequency $\Omega' = k\Delta\Omega$ that has had its *amplitude scaled* by a factor of $|X(\Omega')|$ and has been *time shifted* by an amount that depends on $\arg X(\Omega')$.
- For a given $\Omega' = k\Delta\Omega$ (which is associated with the kth term in the summation), the larger $|X(\Omega')|$ is, the larger the amplitude of its corresponding complex sinusoid $e^{j\Omega' n}$ will be, and therefore the larger the contribution the kth term will make to the overall summation.
- In this way, we can use $|X(\Omega')|$ as a *measure* of how much information a sequence x has at the frequency Ω'.

- The Fourier transform X of the sequence x is referred to as the frequency spectrum of x.
- The magnitude $|X(\Omega)|$ of the Fourier transform X is referred to as the magnitude spectrum of x.
- The argument $\arg X(\Omega)$ of the Fourier transform X is referred to as the phase spectrum of x.
- Since the Fourier transform is a function of a real variable, a sequence can potentially have information at any real frequency.
- Earlier, we saw that for periodic sequences, the Fourier transform can only be nonzero at integer multiples of the fundamental frequency.
- So, the Fourier transform and Fourier series give a consistent picture in terms of frequency spectra.
- Since the frequency spectrum is complex (in the general case), it is *usually represented using two plots*, one showing the magnitude spectrum and one showing the phase spectrum.

- Recall that, for a *real* sequence x, the Fourier transform X of x satisfies

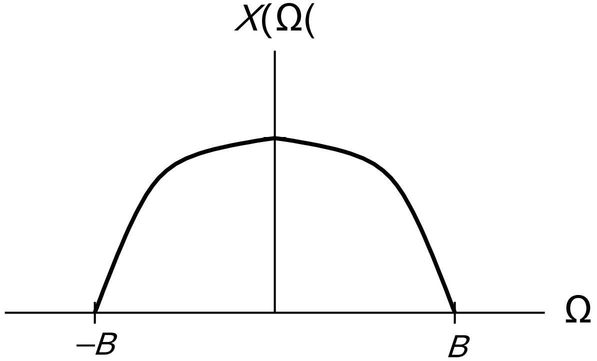
$$X(\Omega) = X^*(-\Omega)$$

(i.e., X is *conjugate symmetric*), which is equivalent to

$$|X(\Omega)| = |X(-\Omega)| \quad \text{and} \quad \arg X(\Omega) = -\arg X(-\Omega).$$

- Since $|X(\Omega)| = |X(-\Omega)|$, the magnitude spectrum of a *real* sequence is always *even*.
- Similarly, since $\arg X(\Omega) = -\arg X(-\Omega)$, the phase spectrum of a *real* sequence is always *odd*.
- Due to the symmetry in the frequency spectra of real sequences, we typically *ignore negative frequencies* when dealing with such sequences.
- In the case of sequences that are complex but not real, frequency spectra do not possess the above symmetry, and *negative frequencies become important*.

- A sequence x with Fourier transform X is said to be bandlimited if, for some nonnegative real constant B, $X(\Omega) = 0$ for all Ω satisfying $|\Omega| > B$.
- In the context of real sequences, we usually refer to B as the bandwidth of the signal x.
- The (real) sequence with the Fourier transform X shown below has bandwidth B.



- One can show that a sequence *cannot be both time limited and bandlimited*. (This follows from the time/frequency scaling property of the Fourier transform.)

Section 10.5

Fourier Transform and LTI Systems

- Consider a LTI system with input x, output y, and impulse response h, and let X, Y, and H denote the Fourier transforms of x, y, and h, respectively.
- Since $y(n) = x * h(n)$, we have that

$$Y(\Omega) = X(\Omega)H(\Omega).$$

- The function H is called the frequency response of the system.
- A LTI system is *completely characterized* by its frequency response H. The
- above equation provides an alternative way of viewing the behavior of a LTI system. That is, we can view the system as operating in the frequency domain on the Fourier transforms of the input and output signals.
- The frequency spectrum of the output is the product of the frequency spectrum of the input and the frequency response of the system.

- In the general case, the frequency response H is a complex-valued function.
- Often, we represent $H(\Omega)$ in terms of its magnitude $|H(\Omega)|$ and argument $\arg H(\Omega)$.
- The quantity $|H(\Omega)|$ is called the magnitude response of the system.
- The quantity $\arg H(\Omega)$ is called the phase response of the system.
- Since $Y(\Omega) = X(\Omega)H(\Omega)$, we trivially have that

$$|Y(\Omega)| = |X(\Omega)|\,|H(\Omega)| \quad \text{and} \quad \arg Y(\Omega) = \arg X(\Omega) + \arg H(\Omega).$$

- The magnitude spectrum of the output equals the magnitude spectrum of the input times the magnitude response of the system.
- The phase spectrum of the output equals the phase spectrum of the input plus the phase response of the system.